

Progressive diagonalization and applications♣

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Abstract. We give a partial review of what is known so far on stability of periodically driven quantum systems versus regularity of the bounded driven force. In particular we emphasize the fact that unbounded degeneracies of the unperturbed Hamiltonian are allowed. Then we give a detailed description of an extension to some unbounded driven forces. This is done by representing the Schrödinger equation in the instantaneous basis of the time dependent Hamiltonian with a method that we call progressive diagonalization.

1 The main theorem

This paper concerns the spectral analysis of Floquet Hamiltonians associated to quantum systems which are periodically driven. They are described by the Schrödinger equation:

$$(-i\partial_t + H_0 + V(\omega t))\psi = 0, \quad \begin{cases} H_0 \text{ selfadjoint in } \mathcal{H}, \\ t \rightarrow V(t), \text{ } 2\pi \text{ periodic,} \\ \omega > 0, \text{ a real frequency,} \\ \mathbb{R} \ni t \rightarrow \psi(t) \in \mathcal{H}, \end{cases} \quad (1)$$

where \mathcal{H} is a separable Hilbert space, and H_0 has the following type of spectral decomposition (E_n, P_n denoting respectively the eigenvalues in ascending order and the eigenprojections)

$$H_0 = \sum_{n=1}^{\infty} E_n P_n, \quad M_n := \dim P_n < \infty$$

with a growing gap condition of the type

$$\exists \sigma > 0, \quad \frac{1}{(\Delta E_\sigma)^\sigma} := \sum_{m \neq n} \frac{M_m M_n}{|E_m - E_n|^\sigma} < \infty. \quad (2)$$

The driven force is given by a time dependent real potential V which is, in the first part of this paper, bounded in the following norm

$$\|V\|_r := \sup_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \|V(k, m, n)\| \max\{|k|^r, 1\}. \quad (3)$$

where $\|V(k, m, n)\|$ denotes the operator norm of

$$V(k, m, n) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} P_m V(t) P_n dt : \mathcal{H} \rightarrow \mathcal{H}. \quad (4)$$

The following main theorem is about the selfadjoint operator $K := K_0 + V$ with $K_0 := -i\omega \partial_t \otimes 1 + 1 \otimes H_0$ acting in the Hilbert space $\mathcal{K} := L^2(S^1) \otimes \mathcal{H}$, i.e. functions which are 2π -periodic in time.

Theorem 1.

Let $\omega_0 > 0$, $\Omega_0 := [\frac{8}{9}\omega_0, \frac{9}{8}\omega_0]$, assume (2) for some $\sigma > 0$ and let

$$\Delta_0 := \min_{m \neq n} |E_m - E_n|.$$

Then, $\forall r > \sigma + \frac{1}{2}$, $\exists C_1 > 0$ and $C_2(\sigma, r) > 0$, such that

$$\|V\|_r < \min \left\{ \frac{4\Delta_0}{C_1}, \frac{\omega_0}{C_1}, \frac{\omega_0}{C_2} \left(\frac{\Delta E_\sigma}{\omega_0} \right)^\sigma \right\} \implies$$

$$\exists \Omega_\infty \subset \Omega_0, \quad \text{with} \quad \frac{|\Omega_\infty|}{|\Omega_0|} \geq 1 - \frac{\|V\|_r}{\frac{\omega_0}{C_2} \left(\frac{\Delta E_\sigma}{\omega_0} \right)^\sigma}$$

so that K is pure point for all $\omega \in \Omega_\infty$. $|\Omega_\star|$ denotes the Lebesgue measure of Ω_\star .

The proof of this theorem and its complement that we state at the end of this section can be found in [DLSV]. This theorem is a result in singular perturbation theory since as this is shown in [DSV] one has

$$(2) \quad \xRightarrow{\text{obviously}} \limsup_{n \rightarrow \infty} E_n = +\infty \quad \xRightarrow{[\text{DSV}]} \quad \forall a.a. \, \omega, \, \text{spect } K_0 = \mathbb{R}$$

i.e. for almost all ω , K_0 has a dense pure point spectrum. To be able to overcome this small divisors difficulty we use a technique which consists in applying to $K_0 + V$ an infinite sequence of unitary transforms so that at the s^{th} step

$$K_0 + V \sim K_0 + G_s + V_s, \quad \text{with} \quad V_s = \mathcal{O}(\|V\|_{r-\sigma-\frac{1}{2}}^{2^{s-1}})$$

i.e. $K_0 + V$ is unitarily equivalent to a diagonal part $K_0 + G_s$, in the eigenbasis of K_0 , plus an off diagonal part V_s which is super exponentially small in the s variable provided $\|V\|_r$ is small enough. This is why we like to call this method *progressive diagonalization* although it is known usually under the name KAM-type method, since this is an adaptation of the famous Kolmogorov-Arnold-Moser method originally invented to treat perturbations of integrable Hamiltonians in classical mechanics.

An extension of the previous theorem to certain classes of unbounded perturbations V is given in section §3, see Theorem 3. We shall do it by (block-) diagonalizing $H_0 + V(t)$ for each t , i.e. by constructing a time

dependent unitary transform $J(t)$ such that $H_0 + V(t) = J(t)(H_0 + G(t))J(t)^*$, where $H_0 + G(t)$ commutes with H_0 , thus

$$K_0 + V \sim -i\omega\partial_t + H_0 + G(t) - i\omega J(t)^* \dot{J}(t)$$

(\dot{J} denotes the time derivative of J). V and H_0 are such that the new perturbation $G(t) - i\omega J(t)^* \dot{J}(t)$ is bounded so that we can apply Theorem 1. This diagonalization of $H_0 + V$ will be done in details with a progressive diagonalization method (P.D.M.), however simpler than the one used for Theorem 1 since we do not have small divisors here. We think it's a good starting point for readers which are not familiar with this P.D.M.. This idea of regularizing an unbounded V by going to the instantaneous basis of $H_0 + V(t)$ is not new, (see e.g. [H3, ADE]). Let us mention the recent work [BaG] which also treats the Schrödinger equation with unbounded perturbations which are quasi-periodic and *analytic* in time; here we treat the *differential* periodic case.

The use of KAM technique to diagonalize quantum Floquet Hamiltonians appeared first in [B] who considered pulsed rotors of the type

$$-i\omega\partial_t + H_0 + f(t)W(x) \quad \text{acting in} \quad L^2(S^1) \otimes L^2(S^d), \quad (5)$$

where $d = 1$, $H_0 = -\Delta$, f and W are analytic. Later on, the adaptation of the Nash-Moser ideas to treat non analytic perturbations was done in [C] for the special case of (one dimensional) driven harmonic oscillators. These ideas were extended to a large class of models in [DS]. However to our knowledge the above Theorem 1 is the first one who allows degeneracies of eigenvalues of H_0 which are not uniformly bounded with respect to the quantum number n . Consequently we can exhibit frequencies such that the quantum top model in arbitrary dimension, i.e. the higher dimensional versions of the pulsed rotor (see (5) and §4.1), is pure point.

One of the main goal of the spectral analysis of these Floquet Hamiltonians is the study of the stability of periodically driven quantum systems since it is known that

$$K_0 + V \text{ is pure point} \stackrel{[EV]}{\iff} \lim_{n \rightarrow \infty} \sup_{t \geq 0} \left\| \sum_{m=n}^{\infty} P_m \psi(t) \right\| = 0, \quad \forall \psi(0) \in \mathcal{H} \quad (6)$$

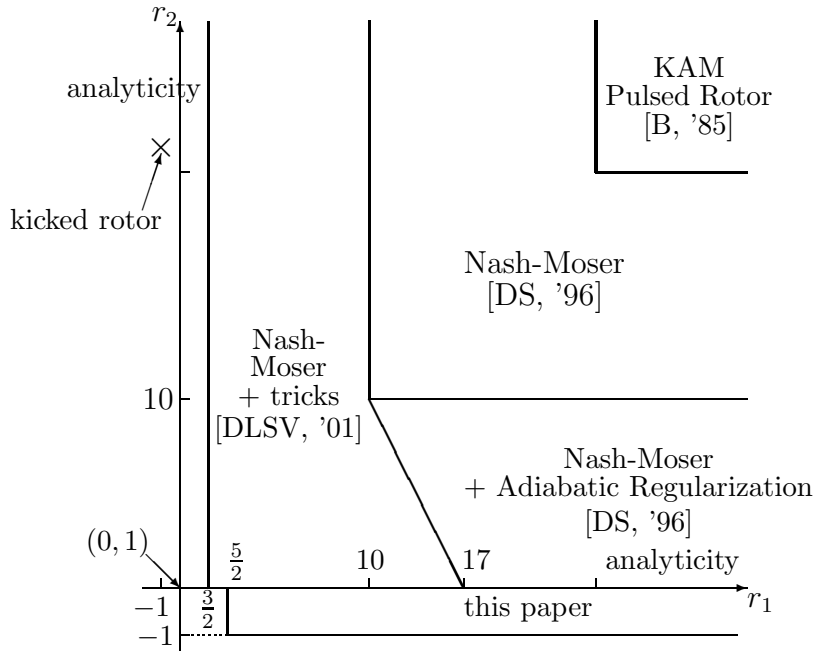
because $\exp(-iT(K_0 + V))$ is unitarily equivalent to $1 \otimes U(T, 0)$ where $U(T, 0)$ denotes the propagator over the period T associated to the Schrödinger equation (1), (see [H1, Y]). The above r.h.s. says that the probability that the quantum trajectory with an arbitrary initial condition $\psi(0)$ explores in the full history the eigenstates of H_0 of energy higher than E_n becomes smaller and smaller as n gets larger and larger. On the other hand if $\psi(0)$ belongs to the continuous spectral subspace of $U(T, 0)$ then (see [EV])

$$\forall m \in \mathbb{N}, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|P_m \psi(t)\| dt = 0$$

which means that in the time average the probability that the trajectory stays in the m^{th} spectral subspace of H_0 vanishes.

$$\|V\|_{r_1, r_2} := \sup_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \|V(k, m, n)\| \langle k \rangle^{r_1} \langle m - n \rangle^{r_2}$$

$\sigma = 1 + 0$, $\sup_n M_n < \infty$, ω non resonant and $\|V\|_{r_1, r_2}$ small enough



The pure pointness of K from which follows the stability (6) does not imply in general that

4

i.e. the uniform boundedness of the energy. Notice that the converse is obviously true. It is believed that to get (7) one should require sufficient regularity of the eigenprojectors of K . That is why the following complement to theorem 1 may be of interest. We have also added some explicit bound on the constants C_1 and C_2 . It will be necessary in §3 to consider potentials V which depend on the frequency ω in a more elaborate way. Suppose that $V : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{H})$ is a bounded measurable function, which is 2π periodic with respect to the first variable and such that for almost all $t \in \mathbb{R}$ and $\omega \in \mathbb{R}_+$, $V(t, \omega)^* = V(t, \omega)$. For such V we modify $\|V\|_r$ as follows:

$$\|V\|_r := \sup_{\omega, \omega' \in \Omega_0} \sup_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}, n \in \mathbb{N}} (\|V_{kmn}(\omega)\| + \omega_0 \|\partial_\omega V_{kmn}(\omega, \omega')\|) \max\{|k|^r, 1\}$$

where

$$V_{kmn}(\omega) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} P_m V(t, \omega) P_n dt$$

and

$$\partial_\omega V_{kmn}(\omega, \omega') := \frac{V_{kmn}(\omega) - V_{kmn}(\omega')}{\omega - \omega'}.$$

Complement to Theorem 1.

In addition to the statements of Theorem 1 one also has

(a) each eigenprojection P of K is bounded in the norm

$$\|P\|_{r-\sigma-\frac{1}{2}} = \sup_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \|P(k, m, n)\| \max\{|k|^{r-\sigma-\frac{1}{2}}, 1\}$$

(b) The following values of the constants are allowed: $C_1 = 24305$, and

$$C_2(\sigma, r) = \frac{C(\sigma)}{\min\{r - \sigma - \frac{1}{2}, \frac{7}{8}(2\sigma + 1)\}^3}, \quad \text{with}$$

$$C(\sigma) = 25223 \pi (2\sigma + 1)^3 \left(\frac{2(2\sigma + 1)}{e(1 - \exp(\frac{-4}{2\sigma+1}))} \right)^{\sigma+\frac{1}{2}}.$$

(c) Theorem 1 extends to $V : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathcal{B}(\mathcal{H})$ of the type described above.

In the progressive diagonalization method one must solve at each step a commutator equation of the type

$$[K_0 + G_s, W_s] = V_s.$$

This is done block-component wise i.e. with the notation (4) solving for each $(k, m, n) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ the following matrix equation in the unknown $W_s(k, m, n)$:

$$(\omega k + E_m + G_s(m))W_s(k, m, n) - W_s(k, m, n)(E_n + G_s(n)) = V_s(k, m, n).$$

We are interested in the best possible estimate of $\|W_s(k, m, n)\|$ in terms of $\|V_s(k, m, n)\|$. In §2 we report on a method to solve this equation which, we believe, is the best one known so far. Finally we present two applications in §4.

2 On the commutator equation

Let E and F be two Hilbert spaces and $\mathcal{B}(E)$, $\mathcal{B}(F)$ the Banach spaces of bounded endomorphisms on E and F respectively, equipped with the usual operator norm. Let $A \in \mathcal{B}(E)$ and $B \in \mathcal{B}(F)$ be selfadjoint and such that

$$d_{A,B} := \text{dist}(\text{spect}(A), \text{spect}(B)) > 0; \quad (8)$$

to each Y in $\mathcal{B}(F, E)$, the bounded homomorphisms from F into E , we want to associate $X \in \mathcal{B}(F, E)$ defined as follows:

$$\text{ad}_{A,B} X = Y, \quad \text{where} \quad \text{ad}_{A,B} X := A X - X B$$

A review on answers about this question can be found in the beautiful paper [BhaRos]. In particular one can find there the following result.

Lemma.

Under the conditions described above $\text{ad}_{A,B}$ is a bounded linear mapping which has a bounded inverse $\Gamma_{A,B}$ and:

$$\|\Gamma_{A,B}\| \leq \frac{\pi}{2} \frac{1}{d_{A,B}}.$$

Remark.

(a) In fact there are some special cases when the constant $\frac{\pi}{2}$ can be replaced by 1. We have not found useful to pay attention to these subtleties here.

(b) The solution X is given by:

$$X := \int_{\mathbb{R}} e^{-itA} Y e^{itB} f(t) dt$$

with any $f \in L^1(\mathbb{R})$ such that its Fourier transform \hat{f} obeys $\sqrt{2\pi} \hat{f}(s) = s^{-1}$ on the set $\text{spect } A - \text{spect } B$. Clearly this shows that $\|X\| \leq \|f\|_1 \|Y\|$. Optimizing over such f leads to the constant $\frac{\pi}{2}$.

3 Unbounded perturbations

3.1 The setting

We start by the description of the class of unbounded perturbations we shall consider. Let H_0 be a *positive selfadjoint* operator on the Hilbert space \mathcal{H} and $\{P_n\}_{n \in \mathbb{N}}$ a complete set of mutually orthogonal projections which reduces H_0 . We denote by $E_n := P_n H_0 P_n = H_0 P_n$, $\mathcal{H}_n := \text{Ran } P_n$ and $\mathcal{H}^{(d)}$ the algebraic direct sum: $\bigoplus_{n \in \mathbb{N}} \text{Ran } P_n$. We introduce the following Banach spaces: for all $1 \leq p \leq \infty$

$$L^p(\mathcal{H}^{(d)}) \ni u = \bigoplus_{n \in \mathbb{N}} u_n : \iff \|u\|_p^p := \sum_{n \in \mathbb{N}} \|u_n\|^p < \infty$$

where $\|\cdot\|$ is the norm of \mathcal{H} . Of course $L^2(\mathcal{H}^{(d)})$ is nothing but \mathcal{H} and $\|u\|_\infty := \sup_n \|u_n\|$.

Then $\mathcal{B}^{q,p}$, $1 \leq p, q \leq \infty$, will denote the Banach spaces of bounded operators defined on $L^p(\mathcal{H}^{(d)})$ with values in $L^q(\mathcal{H}^{(d)})$ and $\|\cdot\|_{q,p}$ its operator norm. We note that

$$\|X\|_{\infty,1} = \sup_{n,m \in \mathbb{N}} \|X(m,n)\|$$

and

$$\|X\|_{1,1} = \sup_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \|X(m,n)\|, \quad \|X\|_{\infty,\infty} = \sup_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} \|X(m,n)\|$$

where $X(m,n)$ is the block element of X which acts from \mathcal{H}_n into \mathcal{H}_m and $\|X(m,n)\|$ its norm as a bounded operator in \mathcal{H} . We shall say that $X \in \mathcal{B}^{q,p}$ is *symmetric* resp. *antisymmetric* if $X(n,m) = X(m,n)^*$ resp. $X(n,m) = -X(m,n)^*$ for all m, n . This definition coincides with the usual one in $\mathcal{B}^{2,2} \sim \mathcal{B}(\mathcal{H})$. We remark that if X is symmetric or antisymmetric then $X \in \mathcal{B}^{1,1}$ if and only if $X \in \mathcal{B}^{\infty,\infty}$ if and only if $X \in \mathcal{B}_{SH} := \mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$; this last operator space is equipped with the norm $\|X\|_{SH} := \max\{\|X\|_{1,1}, \|X\|_{\infty,\infty}\}$. It is known, (see [K, Example III.2.3]) that \mathcal{B}_{SH} is contained in all $\mathcal{B}^{p,p}$, $1 \leq p \leq \infty$, and in particular in $\mathcal{B}(\mathcal{H})$, and it is easy to check that \mathcal{B}_{SH} is a Banach algebra.

On the spectra of H_0 we require the two following conditions:

$$\frac{1}{\Delta E} := \sup_m \sum_{n \neq m} \frac{1}{\Delta_{m,n}} < \infty \quad (\text{GGCH}_0)$$

with

$$\Delta_{m,n} := \text{dist}(\text{spect } E_m, \text{spect } E_n)$$

which expresses that the distances between the spectrum of two blocks E_m and E_n grows sufficiently rapidly with $|m - n|$. The second condition says that each blocks E_n must be bounded:

$$\forall n, \quad E_n \in \mathcal{B}(\mathcal{H}). \quad (\text{BBCH}_0)$$

3.2 A Class of unbounded perturbations

We make the following assumptions on the perturbation of H_0 to be considered:

$$V \in \mathcal{B}^{\infty,1} \text{ and is symmetric.} \quad (\text{UV})$$

Strictly speaking such a V is not in general an operator acting in \mathcal{H} but the following estimate shows that it can be seen as H_0 -bounded in the quadratic form sense with zero relative bound: let $R_0(a) := (H_0 - a)^{-1}$ with $a < 0$ then

$$\|R_0(a)^{\frac{1}{2}} V R_0(a)^{\frac{1}{2}}\| \leq \sum_{n \in \mathbb{N}} \frac{\|V\|_{\infty,1}}{\text{dist}(a, \text{spect } E_n)} \xrightarrow{a \rightarrow -\infty} 0.$$

Indeed since that $R_0(a)^{\frac{1}{2}}$ acts diagonally on $\mathcal{H}^{(d)}$ one gets immediately using (GGCH₀) that

$$\max \left\{ \|R_0(a)^{\frac{1}{2}}\|_{1,2}, \|R_0(a)^{\frac{1}{2}}\|_{2,\infty} \right\} \leq \left(\sum_{m \in \mathbb{N}} \frac{1}{\text{dist}(a, \text{spect } E_m)} \right)^{\frac{1}{2}}.$$

This allows to consider $R_0(a)^{\frac{1}{2}} V R_0(a)^{\frac{1}{2}}$ as

$$L^2(\mathcal{H}^{(d)}) \xrightarrow{R_0(a)^{\frac{1}{2}}} L^1(\mathcal{H}^{(d)}) \xrightarrow{V} L^\infty(\mathcal{H}^{(d)}) \xrightarrow{R_0(a)^{\frac{1}{2}}} L^2(\mathcal{H}^{(d)});$$

hence its above estimate and limiting behaviour as $a \rightarrow \infty$ follow easily.

3.3 Progressive diagonalization of $H_0 + V$

Here we show the

Theorem 2.

Assume $H_0 \geq 0$ and V obey (GGCH₀), (BBCH₀) and (UV). If

$$\|V\|_{\infty,1} \leq \frac{\Delta E}{8}$$

there exists $J \in \mathcal{B}_{\text{SH}}$ and $G \in \mathcal{B}^{2,2}$ such that

$$H_0 + V = J(H_0 + G)J^*$$

with (i) $[H_0, G] = 0$,

(ii) J is unitary in $\mathcal{B}^{2,2}$,

(iii) $\|J\|_{\text{SH}} \leq \frac{3}{2}$ and $\|G\| \leq 2\|V\|_{\infty,1}$

(iv) $[H_0, J] \in \mathcal{B}^{\infty,1}$.

Remark.

(a) Since ΔE is smaller than the smallest gap of H_0 the bound on $\|G\| \leq \frac{\Delta E}{4}$ says in particular that each gap of H_0 remains open after perturbation by V . The bound on J will be used later on.

(b) The algorithm says that G belongs to $\mathcal{B}^{\infty,1}$ which combined with (i) gives $G \in \mathcal{B}^{2,2}$.

(c) The property (iv) is the key of the so-called "adiabatic regularization method" first proposed by Howland [H2] for the case of bounded V . Its proof is immediate from the formula $H_0 + V = J(H_0 + G)J^*$ since it is equivalent to $[H_0, J] = JG - VJ$ and since $J \in \mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$, $G, V \in \mathcal{B}^{\infty,1}$. This trick was systematically used in [DS, §3].

3.3.1 The formal algorithm

With $H_0 + V$ we form a first 4-tuple of operators

$$(U_0 := \text{id}, \quad G_1 := \text{diag } V, \quad V_1 := \text{offdiag } V, \quad H_1 := H_0 + G_1 + V_1)$$

where

$$\text{diag } X := \sum_{n \in \mathbb{N}} P_n X P_n, \quad \text{offdiag } X := \sum_{m \neq n} P_m X P_n$$

Clearly U_0 is unitary, G_1 diagonal (i.e. commutes with H_0), and V_1 is symmetric. Starting from this 4-tuple we generate recursively an infinite sequence of such 4-tuples as follows: let W_s be the solution of

$$[H_0 + G_s, W_s] = V_s \text{ \& } \text{diag } W_s = 0;$$

we shall use the notations $\text{ad}_A B := [A, B] := AB - BA$. Then we define

$$H_{s+1} := e^{W_s} H_s e^{-W_s} = H_0 + G_s + \sum_{k=1}^{\infty} \frac{k}{(k+1)!} \text{ad}_{W_s}^k V_s \quad (9)$$

and set

$$U_s := e^{W_s} U_{s-1}, \quad G_{s+1} := \text{diag } H_{s+1} - H_0, \quad V_{s+1} = \text{offdiag } H_{s+1}.$$

Since $H_0 + G_s$ and V_s are symmetric W_s is antisymmetric and therefore e^{W_s} and U_s are formally unitary. Consequently

$$H_0 + G_{s+1} + V_{s+1} = U_s (H_0 + V) U_s^{-1} \quad (10)$$

and to achieve our goal we have to prove that $V_s \rightarrow 0$, $G_s \rightarrow G_\infty$ and $U_s \rightarrow U_\infty$ as $s \rightarrow \infty$.

3.3.2 Convergence of the algorithm

We solve the commutator equation $[H_0 + G_s, W_s] = V_s$ block wise, i.e. for all $m \neq n$, we look for $W_s(m, n)$ such that

$$(E_m + G_s(m))W_s(m, n) - W_s(m, n)(E_n + G_s(n)) = V_s(m, n).$$

Notice the notation $G_s(m) := G_s(m, m)$. Assume for the moment that

$$\forall s \geq 1, \forall m \in \mathbb{N}, \quad 4\|G_s(m)\| \leq \Delta_m := \inf_{m \neq n} \Delta_{m,n} \quad (11)$$

this implies that $H_0 + G_s$ fulfills (BBCH₀) and

$$\forall m \neq n, \quad \text{dist}(\text{spect } E_m + G_s(m), \text{spect } E_n + G_s(n)) \geq \frac{1}{2} \Delta_{m,n}.$$

Hence by the lemma of §2 we know that $W_s(m, n)$ is well defined and obeys

$$\|W_s(m, n)\| \leq \pi \frac{\|V_s\|_{\infty,1}}{\Delta_{m,n}} \quad \Rightarrow \quad \|W_s\|_{SH} \leq \pi \frac{\|V_s\|_{\infty,1}}{\Delta E}$$

i.e. W_s belongs to $\mathcal{B}^{1,1} \cap \mathcal{B}^{\infty,\infty}$. This shows that $\text{ad}_{W_s} : \mathcal{B}^{\infty,1} \rightarrow \mathcal{B}^{\infty,1}$ is bounded by $2\pi\|V_s\|_{\infty,1}\Delta E^{-1}$ and due to (9)

$$\|V_{s+1}\|_{\infty,1} \leq \Phi\left(\frac{2\pi\|V_s\|_{\infty,1}}{\Delta E}\right) \|V_s\|_{\infty,1}$$

where $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the strictly increasing analytic function defined by $\Phi(x) := e^x - \frac{1}{x}(e^x - 1)$ whose Taylor expansion is $\sum_{k \geq 1} (k/(k+1)!) x^k$.

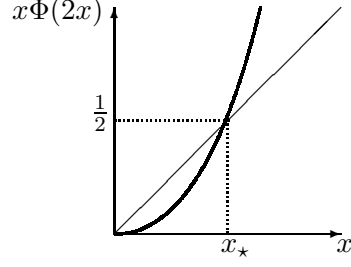


Figure 2. Graph of $x \rightarrow x\Phi(2x)$ and its fix point $x_* = \frac{1}{2}$.

With $x_s := \pi \|V_s\|_{\infty,1} \Delta E^{-1}$, the above inequality becomes $x_{s+1} \leq \Phi(2x_s)x_s$. This is an elementary exercise to check that the series $\{x_s\}_s$ is summable if $x_1 < x_* := 1/2$. Thus we get

$$\|V\|_{\infty,1} \leq \frac{\Delta E}{8} \quad \Rightarrow \quad \sum_{s=1}^{\infty} \|W_s\|_{\text{SH}} \leq \sum_{s=1}^{\infty} x_s \leq \frac{x_1}{1 - \Phi(2x_1)}$$

The summability of $\{x_s\}_s$ implies that $\|V_s\|_{\infty,1} \rightarrow 0$ as $s \rightarrow \infty$ and that $\sum_{s \geq 1} \|W_s\|_{\text{SH}} < \infty$; this last property shows that U_s is convergent in \mathcal{B}_{SH} to some U_∞ as $s \rightarrow \infty$.

We must check now whether the required property on the G_s , i.e. (11), is verified. Since $G_{s+1} - G_s = \text{diag } \Phi(\text{ad}_{W_s})V_s$ and that $\Delta_m > \Delta E$ we have successively

$$\begin{aligned} (11) \quad &\Leftrightarrow \sum_{s=1}^{\infty} \|G_{s+1} - G_s\| + \|G_1\| \leq \frac{1}{4} \Delta_m \\ &\Leftrightarrow \sum_{s=1}^{\infty} x_s \Phi(2x_s) \frac{1}{\pi} \Delta E + \|G_1\| \leq \frac{1}{4} \Delta E \\ &\Leftrightarrow \|G_1\| \leq 0.13 \Delta E \quad \Leftarrow \quad \|V\|_{\infty,1} \leq \frac{\Delta E}{8} \end{aligned}$$

since one can check numerically that $\frac{1}{4} - \frac{1}{\pi} \sum_{s=1}^{\infty} x_s \Phi(2x_s) \geq 0.13$ if $x_1 \leq \pi/8$ (see below for this bound on x_1). Thus (11) is true and we have also shown that G_s converges to some diagonal and bounded G_∞ as $s \rightarrow \infty$.

To pass from (10) to $H_0 + G_\infty = U_\infty(H_0 + V)U_\infty^{-1}$ using the three ingredients $\|V_s\|_{\infty,1} \rightarrow 0$, $G_s \rightarrow G_\infty$ and $U_s \rightarrow U_\infty$ is not as obvious as it seems; we have to adapt the technique of [DS, §2.4]. We have renamed G_∞ by G and U_∞ by J for later convenience.

Finally we derive the bound on $\|U_\infty\|_{\text{SH}}$ and $\|G_\infty\|$.

$$\|U_\infty\|_{\text{SH}} \leq \exp\left(\sum_{k=1}^{\infty} \|W_k\|_{\text{SH}}\right) \leq \frac{3}{2}$$

since one can check numerically that $\exp(\sum_{s=1}^{\infty} x_s \Phi(2x_s)) \leq \frac{3}{2}$ with

$$x_1 := \frac{\pi \|V_1\|_{\infty,1}}{\Delta E} \leq \frac{\pi}{8}.$$

Concerning G_s notice that $\|X\|_{\text{SH}} = \|X\|$ if X is diagonal; then

$$\|G_\infty\| \leq \|G_1\| + \sum_{s=1}^{\infty} \|G_{s+1} - G_s\| \leq \|V\|_{\infty,1} + \sum_{s=1}^{\infty} x_s \Phi(2x_s) \frac{1}{\pi} \Delta E$$

$$\begin{aligned}
&\leq \|V\|_{\infty,1} + \left(\Phi(2x_1) + \frac{\Phi(2x_1)\Phi(2x_1\Phi(2x_1))}{1 - \Phi(2x_1\Phi(2x_1)\Phi(2x_1\Phi(2x_1)))} \right) x_1 \frac{\Delta E}{\pi} \\
&= \|V\|_{\infty,1} \left(1 + \Phi(2x_1) + \frac{\Phi(2x_1)\Phi(2x_1\Phi(2x_1))}{1 - \Phi(2x_1\Phi(2x_1)\Phi(2x_1\Phi(2x_1)))} \right) \\
&\leq 2\|V\|_{\infty,1}.
\end{aligned}$$

The above analytic bound on $\sum_{s=1}^{\infty} x_s \Phi(2x_s)$ is obtained with elementary manipulation and we end up with a numerical computation with $x_1 = \frac{\pi}{8}$ (notice that Φ is increasing).

3.4 Pure pointness of $K_0 + V$

Let $V : \mathbb{R} \rightarrow \mathcal{B}^{\infty,1}$ be a 2π -periodic symmetric function, with the notation (4) we define the new norm

$$\|V\|_r := \sup_{m,n \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \|V(k, m, n)\| \max\{|k|^r, 1\}. \quad (12)$$

We shall prove that $K := K_0 + V$ is selfadjoint on a suitable domain and

Theorem 3.

Let $\omega_0 > 0$, $\Omega_0 := [\frac{8}{9}\omega_0, \frac{9}{8}\omega_0]$ assume (2) for some $\sigma > 0$ and let $\Delta_0 := \min_{m \neq n} |E_m - E_n|$. Then, $\forall r > \sigma + \frac{3}{2}$, $\exists C_1 > 0$ and $\tilde{C}_2(\sigma, r) > 0$, such that

$$\|V\|_r < \frac{1}{2(1 + 8\frac{\omega_0}{\Delta E})} \min \left\{ \frac{4\Delta_0}{C_1}, \frac{\omega_0}{C_1}, \frac{\omega_0}{\tilde{C}_2} \left(\frac{\Delta E_{\sigma}}{\omega_0} \right)^{\sigma}, 2^{-r} \frac{\Delta E + 8\omega_0}{4} \right\}$$

implies

$$\exists \Omega_{\infty} \subset \Omega_0, \quad \text{with} \quad \frac{|\Omega_{\infty}|}{|\Omega_0|} \geq 1 - \frac{2(1 + 8\frac{\omega_0}{\Delta E})\|V\|_r}{\frac{\omega_0}{\tilde{C}_2} \left(\frac{\Delta E_{\sigma}}{\omega_0} \right)^{\sigma}}$$

so that K is pure point for all $\omega \in \Omega_{\infty}$.

In addition one also has that each eigenprojection P of K is bounded in the norm

$$\|P\|_{r-\sigma-\frac{3}{2}} = \sup_{m \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \|P(k, m, n)\| \max\{|k|^{r-\sigma-\frac{3}{2}}, 1\}$$

and $\tilde{C}_2(\sigma, r) = C_2(\sigma, r-1)$, where C_1 and C_2 are the constants of Theorem 1.

Proof. (a) As said in the introduction the strategy consists in proving that $H_0 + V(t) = J(t)(H_0 + G(t))J^*(t)$ using Theorem 2 for each t , then

$$K_0 + V = -i\omega\partial_t + H_0 + V = J(-i\omega\partial_t + H_0 + \tilde{V})J^*$$

where $\tilde{V}(t) := G(t) - i\omega J^*(t)\dot{J}(t)$ will be seen to fulfill Theorem 1.

(b) Selfadjointness of $K_0 + V$ is not an easy matter since the quadratic form technique cannot be used here because K_0 is not bounded below. We shall establish it indirectly. First with the P.D.M. we shall get the existence of the strongly C^1 map $J : S^1 \rightarrow \mathcal{B}_{\text{SH}}$ such that $1 \otimes H_0 + V = J(1 \otimes H_0 + G)J^*$.

Then it is easily verified that $K_0 + V$ is selfadjoint on $J\text{dom } K_0$ since \tilde{V} is bounded.

(c) Let $w_r(k) := 2^r \max\{|k|^r, 1\}$ for some $r \geq 0$, we shall use the notations

$$w_r V := \{w_r(k)V(k, m, n), k \in \mathbb{Z}, m, n \in \mathbb{N}\};$$

it is straightforward to check that

$$\begin{aligned}\mathcal{E}_r &:= \{V : S^1 \rightarrow \mathcal{B}^{\infty,1}, \|w_r V\|_0 < \infty\} \\ \mathcal{A}_r &:= \{V : S^1 \rightarrow \mathcal{B}^{\infty,1}, \|w_r V\|_0 < \infty\}\end{aligned}$$

are respectively a Banach space and a Banach algebra, with $\mathcal{A}_r \subset \mathcal{E}_r$ and

$$\mathcal{A}_r \mathcal{E}_r \subset \mathcal{E}_r \quad \text{and} \quad \mathcal{E}_r \mathcal{A}_r \subset \mathcal{E}_r.$$

We simply follow §3.3 with \mathcal{H} , H_0 , V , $\mathcal{B}^{\infty,1}$ and \mathcal{B}_{SH} replaced respectively by \mathcal{K} , $1 \otimes H_0$, $S^1 \ni t \rightarrow V(t)$, \mathcal{E}_r and \mathcal{A}_r so that we get as for Theorem 2:

if $\|w_r V\|_0 \leq \Delta E/8$ there exists $J \in \mathcal{A}_r$ and $G \in \mathcal{E}_r$ such that $1 \otimes H_0 + V = J(1 \otimes H_0 + G)J^$ together with*

$$\|w_r J\|_0 \leq \frac{3}{2} \quad \text{and} \quad \|w_r G\|_0 \leq 2\|w_r V\|_0.$$

Therefore $\|w_{r-1} J\|_0 \leq 3/4$ and $\|w_{r-1} G\|_0 \leq \|w_r V\|_0$ since $w_{r-1} \leq w_r/2$. Of course it follows that $\|w_{r-1} J^*\|_0 \leq 3/4$.

It remains to estimate $\|w_r J\|_0$. One has with $J = \prod_{s=1}^{\infty} e^{W_s}$ and $x_1 := \pi\|w_r V\|_0 \Delta E^{-1} \leq \pi/8$:

$$\begin{aligned}\|w_{r-1} J\|_0 &\leq \sum_{s=1}^{\infty} \|w_{r-1} \dot{W}_s\|_0 \exp\left(\sum_{s=1}^{\infty} \|w_{r-1} W_s\|_0\right) \\ &= \frac{1}{2} \sum_{s=1}^{\infty} \|w_r W_s\|_0 \exp\left(\frac{1}{2} \sum_{s=1}^{\infty} \|w_r W_s\|_0\right) \\ &\leq \frac{1}{2} \frac{x_1}{1 - \Phi(2x_1)} \exp\left(\frac{1}{2} \frac{x_1}{1 - \Phi(2x_1)}\right) \\ &\leq \pi \frac{\|w_r V\|_0}{\Delta E} 3 = 3\pi \frac{\|w_r V\|_0}{\Delta E}\end{aligned}$$

since one can check numerically that $(2(1 - \Phi(2x_1)))^{-1} \exp(x_1/(2(1 - \Phi(2x_1))))$ is less than 3 if $x_1 \leq \pi/8$.

Thus we have obtained for all $\omega \in \Omega_0$

$$\begin{aligned}2^{r-1} \|\tilde{V}\|_{r-1} &= \|w_{r-1} \tilde{V}\|_0 \leq \|w_{r-1} G\|_0 + \frac{9}{8} \omega_0 \|w_{r-1} J^*\| \|w_{r-1} J\| \\ &\leq \left(1 + \frac{9}{8} 3\pi \frac{\omega_0}{\Delta E}\right) \|w_r V\|_0 \leq \left(1 + 8 \frac{\omega_0}{\Delta E}\right) 2^r \|V\|_r.\end{aligned}$$

Finally we apply Theorem 1 to $K_0 + \tilde{V}$ with r replaced by $r - 1$ and $\|V\|_r$ by $2\left(1 + 8 \frac{\omega_0}{\Delta E}\right) \|V\|_r$. We also have to impose the additional condition $\|w_r V\|_0 \leq \Delta E/8$. ■

4 Applications

4.1 The d dimensional quantum top

Here we give an example of Theorem 1 with unbounded multiplicities of the spectrum of H_0 . We consider the model (5). H_0 is the Laplace-Beltrami operator on the d -dimensional sphere S^d . Then the n^{th} eigenvalue obeys

$$E_n = n(n + d - 1) \quad \text{with} \quad M_n = \binom{n + d}{d} - \binom{n + d - 2}{d} \underset{n \rightarrow \infty}{\sim} \frac{2n^{d-1}}{(d-1)!}$$

so that the growing gap condition (2) is fulfilled if and only if

$$\sum_{m>n} \frac{(mn)^{d-1}}{(m^2 - n^2)^\sigma} < \infty \iff \sigma > 2d - 1.$$

If $f \in C^s(\mathbb{R})$ and $W \in C^u(S^d)$ with

$$s > r + 1 > \sigma + \frac{1}{2} + 1 > 2d + \frac{1}{2} \quad \text{and} \quad u \geq 4$$

Theorem 1 applies (see [DLSV] for details). This model has already been studied by Nenciu in [N] who found a sufficient condition to rule out the absolutely continuous spectrum. We have gathered in the next picture what is known so far concerning this model.

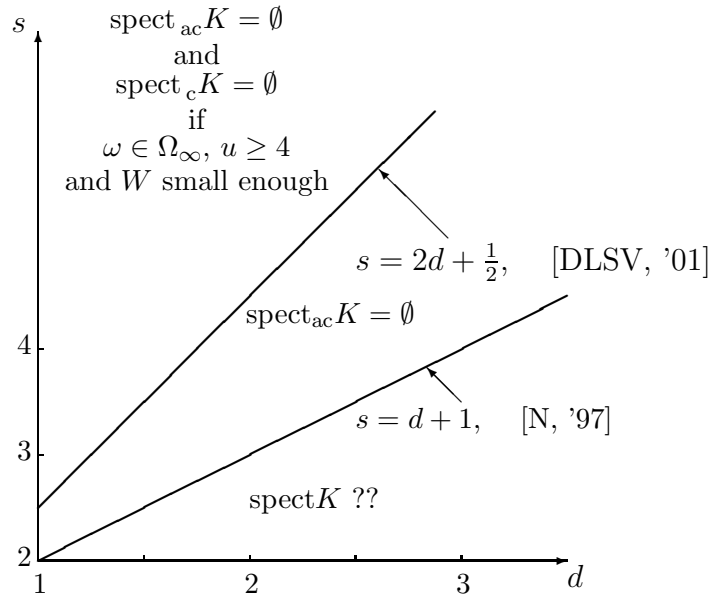


Figure 3. About the quantum top

4.2 The pulsed rotor with a δ point interaction

As an application of Theorem 3 we shall consider the pulsed rotor (5) with $f \in C^s(S^1)$ and W the delta point interaction located at 0. We recall that

this is the interaction associated to the quadratic form on $L^2(S^1)$ defined by $u \rightarrow |u(0)|^2$. One has for the n^{th} eigenprojection of H_0

$$P_n = (\cdot, \varphi_{-n})\varphi_{-n} + (\cdot, \varphi_n)\varphi_n, \quad \text{with} \quad \varphi_n(x) := \frac{1}{\sqrt{2\pi}}e^{inx}$$

except for $P_0 = (\cdot, \varphi_0)\varphi_0$. (UV) is true since $\|\delta\|_{\infty,1} = \pi^{-1}$ because

$$\begin{aligned} \|P_m\delta P_n\| &= \left\| \frac{1}{2\pi} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\| = \frac{1}{\pi} \quad m, n \neq 0 \\ \|P_m\delta P_0\| &= \left\| \frac{1}{2\pi} \begin{pmatrix} 1 & 1 \end{pmatrix} \right\| = \frac{1}{\sqrt{2\pi}} \quad m \neq 0 \\ \|P_0\delta P_0\| &= \frac{1}{2\pi} \end{aligned}$$

Moreover

$$\frac{1}{\Delta E} = \sup_{m \in \mathbb{N}} \sum_{\mathbb{N} \ni n \neq m} \frac{1}{|m^2 - n^2|} = \frac{7}{4} \quad \Rightarrow \quad (\text{GGCH}_0)$$

$$\|E_n\| = \left\| n^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| = n^2 < \infty \quad \Rightarrow \quad (\text{BBCH}_0).$$

Let

$$f(t) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikt}$$

be the Fourier expansion of f . Then

$$\|fW\|_r \leq \frac{1}{\pi} \sum_{k \in \mathbb{Z}} |\hat{f}_k| \max\{|k|^r, 1\}.$$

Since the eigenvalues of H_0 are $\{n^2\}_{n \in \mathbb{N}}$ one has that every $\sigma > 1$ will insure that $\Delta E_\sigma < \infty$. Thus in order to apply Theorem 3 one needs $r > \sigma + 3/2$ i.e. $r > 5/2$ and finally $s > 7/2$ to insure that $\|fW\|_r$ is finite. We have proven that

Let $f \in C^s(\mathbb{R}, \mathbb{R})$ be a 2π - periodic function with $s > 7/2$ and g a real constant. The Floquet operator associated to the time dependent Schrödinger operator $-\Delta + gf(\omega t)\delta(x)$ on $L^2(S^1)$ is pure point provided g is small enough and for appropriate frequencies ω . In such conditions this quantum system is stable in the sense equation (6).

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